

HECKE-CLIFFORD ALGEBRAS AND SPIN HECKE ALGEBRAS III: THE TRIGONOMETRIC TYPE

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ABSTRACT. The notion of trigonometric spin double affine Hecke algebras (tsDaHa) and trigonometric double affine Hecke-Clifford algebras (tDaHCa) associated to classical Weyl groups are introduced. The PBW basis property is established. An algebra isomorphism relating tDaHCa to tsDaHa is obtained.

1. INTRODUCTION

1.1. For an irreducible finite Weyl group W associated to a root system R , there are corresponding affine Weyl group W^a and extended affine Weyl group W^e attached with an affine root system \tilde{R} . There is an interesting family of algebras attached to W^a or W^e , namely the trigonometric double affine Hecke algebra (tDAHA), $\check{H}_{t,c}$, where t, c are certain parameters. The tDAHA is not only a degeneration of the double affine Hecke algebra (DAHA), but also an extension of the degenerate affine Hecke algebra (AHA) as defined by Lusztig in [Lu]. The tDAHA was introduced and studied by Cherednik with application to harmonic analysis and Macdonald polynomials, see [Ch1, Ch2]. Furthermore, when one specializes $t = 0$, the algebra $\check{H}_{t=0,c}$ has a large center, in particular, $\check{H}_{t=0,c}$ is a finite module over its center, and it has interesting connections with algebraic geometry (see [Ob]).

In 1911, I. Schur developed a theory of spin (projective) representations of the symmetric group S_n . We note that studying the spin representations of S_n is equivalent to study the representations of the *spin* symmetric group algebra $\mathbb{C}S_n^-$. For the symmetric group S_n , there is a standard procedure to construct the associated Hecke algebras. Wang [W] has raised the question of whether or not a notion of Hecke algebras associated to the *spin* symmetric group algebra $\mathbb{C}S_n^-$ exists, and has provided a natural construction of the trigonometric DAHA associated to the algebra $\mathbb{C}S_n^-$, denoted by \check{H}_{tr}^- . Moreover, in [W, KW1, KW2], we have developed a theory of spin Hecke algebras of (degenerate) affine and rational double affine type (also cf. [Naz]).

1.2. This paper is a sequel to [W, KW1, KW2]. The main goal here is to construct two classes of algebras which are closely related to the tDAHA associated to each classical finite Weyl group W called the *trigonometric*

double affine Hecke-Clifford algebra (tDaHCa), \ddot{H}_W^ϵ , and the *trigonometric spin double affine Hecke algebra* (tsDaHa), \ddot{H}_W^- . We establish a few basic properties of the algebras \ddot{H}_W^ϵ and \ddot{H}_W^- , including their PBW basis properties. In addition, we prove that for type A case, the algebra \ddot{H}_W^ϵ (respectively, \ddot{H}_W^-) is isomorphic to $\ddot{\mathfrak{H}}_{tr}^\epsilon$ (respectively, to $\ddot{\mathfrak{H}}_{tr}^-$) introduced in [W] in a very different presentation.

1.3. In Section 2, we recall the definition of an extended affine Weyl group W^e for each classical type, and then formulate the corresponding *spin* extended affine Weyl group algebra $\mathbb{C}W^{e-}$. We first start with a finite Weyl group W , and then consider a double cover \widetilde{W} of W associated to a distinguished 2-cocycle as in [KW1, Mo]:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1.$$

We then define a covering of the extended affine Weyl group \widetilde{W}^e of W^e . As a result, we obtain the *spin* extended affine Weyl group algebra $\mathbb{C}W^{e-}$ which is defined to be a certain quotient algebra of $\mathbb{C}\widetilde{W}^e$.

In Section 3, recalling that the Weyl group W acts as automorphisms on the Clifford algebra \mathcal{C}_n , we then extend to an action of W^e on \mathcal{C}_n . We establish a (super)algebra isomorphism

$$\Phi : \mathcal{C}_n \rtimes \mathbb{C}W^e \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^{e-}$$

which is an extended affine analogue to the isomorphism $\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^-$ in [KW1] which goes back to Sergeev [Ser] and Yamaguchi [Yam] for type A.

In Section 4 we introduce the tDaHCa \ddot{H}_W^ϵ . We establish the PBW basis properties for the algebras \ddot{H}_W^ϵ :

$$\ddot{H}_W^\epsilon \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C}_n \otimes \mathbb{C}W^e$$

where $\mathbb{C}[\mathfrak{h}^*]$ denotes the polynomial algebra.

In Section 5, we introduce the tsDaHa \ddot{H}_W^- . The tsDaHa are not only the counterpart of the algebras \ddot{H}_W^ϵ , but also the generalizations of the (degenerate) spin affine Hecke algebras associated to the *spin* group algebras $\mathbb{C}W^-$ in [W, KW1]. Denote by $\mathbb{C}[\mathfrak{h}^*]$ a noncommutative skew-polynomial algebra, we establish the PBW basis properties for \ddot{H}_W^- :

$$\ddot{H}_W^- \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W^{e-}.$$

In addition, we establish a (super)algebra isomorphism:

$$\Phi : \ddot{H}_W^\epsilon \xrightarrow{\cong} \mathcal{C}_n \otimes \ddot{H}_W^-$$

which extends the isomorphism $\Phi : \mathcal{C}_n \rtimes \mathbb{C}W^e \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^{e-}$.

1.4. The constructions of our algebras are canonical, but we have to do it case-by-case since the algebras in a way rely on a choice of orthonormal basis of \mathfrak{h} . However, we hope that the detailed presentations on extended affine Weyl groups could be helpful to the reader.

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2. (SPIN) EXTENDED AFFINE WEYL GROUPS

2.1. **The Weyl group W .** Let W be an (irreducible) finite Weyl group of classical type with the following presentation:

$$\langle s_1, \dots, s_n | (s_i s_j)^{m_{ij}} = 1, m_{ii} = 1, m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{ for } i \neq j \rangle. \quad (2.1)$$

The integers m_{ij} are specified by the Coxeter-Dynkin diagrams whose vertices correspond to the generators of W below. By convention, we only mark the edge connecting i, j with $m_{ij} \geq 4$. We have $m_{ij} = 3$ for $i \neq j$ connected by an unmarked edge, and $m_{ij} = 2$ if i, j are not connected by an edge.

$$\begin{array}{ll}
 A_n & \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \\
 & 1 \quad 2 \quad \quad \quad n-1 \quad n \\
 \\
 B_n(n \geq 2) & \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---}^4 \circ \\
 & 1 \quad 2 \quad \quad \quad n-1 \quad n \\
 \\
 D_n(n \geq 4) & \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \begin{array}{l} \nearrow \circ n \\ \searrow \circ n-1 \end{array} \\
 & 1 \quad 2 \quad \quad \quad n-3 \quad n-2
 \end{array}$$

Denote by $W_{A_{n-1}}$ (respectively, W_{B_n} and W_{D_n}) the finite Weyl group of type A_{n-1} (respectively, B_n and D_n). Then the group W_{D_n} is generated by s_1, \dots, s_n , subject to the following relations:

$$s_i^2 = 1 \quad (i \leq n-1) \quad (2.2)$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i \leq n-2) \quad (2.3)$$

$$s_i s_j = s_j s_i \quad (|i-j| > 1, i, j \neq n) \quad (2.4)$$

$$s_i s_n = s_n s_i \quad (i \neq n-2) \quad (2.5)$$

$$s_{n-2} s_n s_{n-2} = s_n s_{n-2} s_n, \quad s_n^2 = 1. \quad (2.6)$$

Recall $S_n = W_{A_{n-1}}$ is generated by s_1, \dots, s_{n-1} subject to the relations (2.2–2.4) above.

The group W_{B_n} is generated by s_1, \dots, s_n , subject to the defining relation for S_n on s_1, \dots, s_{n-1} and the following additional relations:

$$s_i s_n = s_n s_i \quad (1 \leq i \leq n-2) \quad (2.7)$$

$$(s_{n-1} s_n)^4 = 1, \quad s_n^2 = 1. \quad (2.8)$$

2.2. The extended affine Weyl group W^e . In this subsection, we recall the definitions of the extended affine Weyl groups of classical type. For a more detailed expository, consult [Kir]. Define the set O_W^* by

$$O_W^* = \begin{cases} \{1\}, & \text{if } W = W_{A_{n-1}} \text{ or } W = W_{B_n} \\ \{n\}, & \text{if } W = W_{D_{2k+1}} \\ \{1, n\}, & \text{if } W = W_{D_{2k}}. \end{cases}$$

The extended affine Weyl group W^e is the group generated by s_i 's and $\pi_r^{\pm 1}$ for $r \in O_W^*$. The generators s_i 's obey the relations in (2.1) and the defining relations involving π_r are shown below:

if $W = W_{A_{n-1}}$, then

$$\begin{aligned} \pi_1^2 s_{n-1} &= s_1 \pi_1^2, \\ \pi_1^n s_i &= s_i \pi_1^n \quad (1 \leq i \leq n-1), \\ \pi_1 s_i &= s_{i+1} \pi_1 \quad (1 \leq i \leq n-2); \end{aligned} \quad (2.9)$$

if $W = W_{B_n}$, then

$$\begin{aligned} \pi_1^2 &= 1, \\ \pi_1 s_i &= s_i \pi_1 \quad (2 \leq i \leq n), \\ \pi_1 s_1 \pi_1 s_1 &= s_1 \pi_1 s_1 \pi_1; \end{aligned} \quad (2.10)$$

if $W = W_{D_n}$ and n is odd, we have

$$\begin{aligned} \pi_n^4 &= 1, \\ \pi_n^2 s_{n-1} &= s_n \pi_n^2, \\ \pi_n s_i &= s_{n-i} \pi_n \quad (1 \leq i \leq n-2), \\ \pi_n s_n &= s_1 \pi_n; \end{aligned} \quad (2.11)$$

if $W = W_{D_n}$ and n is even, we have

$$\begin{aligned} \pi_1^2 &= 1, \quad \pi_n^2 = 1, \\ \pi_1 \pi_n &= \pi_n \pi_1, \quad \pi_1 s_1 \pi_1 = \pi_n s_n \pi_n, \\ \pi_1 s_i &= s_i \pi_1, \quad \pi_n s_i = s_{n-i} \pi_n \quad (2 \leq i \leq n-2), \\ \pi_1 s_{n-1} &= s_n \pi_1, \quad \pi_n s_1 = s_{n-1} \pi_n. \end{aligned} \quad (2.12)$$

Remark 2.1. We may define

$$s_0 := \begin{cases} \pi_1 s_{n-1} \pi_1^{-1}, & \text{for } W = W_{A_{n-1}} \\ \pi_1 s_1 \pi_1^{-1}, & \text{for } W = W_{B_n} \text{ or } W = W_{D_{2k}} \\ \pi_n s_{n-1} \pi_n^{-1}, & \text{for } W = W_{D_{2k+1}}. \end{cases}$$

Then the elements s_i 's for $i \geq 0$ generate the affine Weyl group W^a .

2.3. A covering of an extended affine Weyl group. We note here that the Schur multipliers for finite Weyl groups W (and actually for all finite Coxeter groups) have been computed by Ihara and Yokonuma [IY] (also cf. [Kar]). The explicit generators and relations for the corresponding covering groups of W can be found in Karpilovsky [Kar, Table 7.1].

In particular, there is a distinguished double covering \widetilde{W} of W :

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1.$$

We denote by $\mathbb{Z}_2 = \{1, z\}$, and by \tilde{t}_i a fixed preimage of the generators s_i of W for each i . The group \widetilde{W} is generated by z, \tilde{t}_i 's with relations

$$z^2 = 1, \quad (\tilde{t}_i \tilde{t}_j)^{m_{ij}} = \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ z, & \text{if } m_{ij} = 2, 4. \end{cases} \quad (2.13)$$

Let W be $W_{A_{n-1}}$, W_{B_n} or W_{D_n} . We define a *covering of the extended affine Weyl group* W^e , denoted by \widetilde{W}^e , to be the group generated by z, \tilde{t}_i 's, and $\tilde{\pi}_r^{\pm 1}$ ($r \in O_W^*$) such that z is central, z, \tilde{t}_i 's satisfy (2.13), and the following additional relation:

if $W = W_{A_{n-1}}$, set

$$\begin{aligned} \tilde{\pi}_1^2 \tilde{t}_{n-1} &= \tilde{t}_1 \pi_1^2, \\ \tilde{\pi}_1^n \tilde{t}_i &= \tilde{t}_i \tilde{\pi}_1^n, \\ \tilde{\pi}_1 \tilde{t}_i &= z^{n-1} \tilde{t}_{i+1} \tilde{\pi}_1 \quad (1 \leq i \leq n-2); \end{aligned} \quad (2.14)$$

if $W = W_{B_n}$, set

$$\begin{aligned} \tilde{\pi}_1^2 &= 1, \\ \tilde{\pi}_1 \tilde{t}_i &= z \tilde{t}_i \tilde{\pi}_1 \quad (2 \leq i \leq n), \\ \tilde{\pi} \tilde{t}_1 \tilde{\pi}_1 \tilde{t}_1 &= z \tilde{t}_1 \tilde{\pi}_1 \tilde{t}_1 \tilde{\pi}_1; \end{aligned} \quad (2.15)$$

if $W = W_{D_n}$ and n is odd, set

$$\begin{aligned} \tilde{\pi}_n^4 &= z, \\ \tilde{\pi}_n^2 \tilde{t}_{n-1} &= \tilde{t}_n \tilde{\pi}_n^2, \\ \tilde{\pi}_n \tilde{t}_i &= z^{\frac{n-1}{2}} \tilde{t}_{n-i} \tilde{\pi}_n \quad (1 \leq i \leq n-2), \\ \tilde{\pi}_n \tilde{t}_n &= z^{\frac{n-1}{2}} \tilde{t}_1 \tilde{\pi}_n; \end{aligned} \quad (2.16)$$

if $W = W_{D_n}$ and n is even, set

$$\begin{aligned}\tilde{\pi}_1^2 &= 1, \quad \tilde{\pi}_n^2 = z^{\frac{n}{2}+1}, \\ \tilde{\pi}_1 \tilde{\pi}_n &= z \tilde{\pi}_n \tilde{\pi}_1, \quad \tilde{\pi}_1 \tilde{t}_1 \tilde{\pi}_1 = z \tilde{\pi}_n \tilde{t}_n \tilde{\pi}_n, \\ \tilde{\pi}_1 \tilde{t}_i &= \tilde{t}_i \tilde{\pi}_1, \quad \tilde{\pi}_n \tilde{t}_i = z^{\frac{n}{2}} \tilde{t}_{n-i} \tilde{\pi}_n \quad (2 \leq i \leq n-2), \\ \tilde{\pi}_1 \tilde{t}_{n-1} &= \tilde{t}_n \tilde{\pi}_1, \quad \tilde{\pi}_n \tilde{t}_1 = z^{\frac{n}{2}} \tilde{t}_{n-1} \tilde{\pi}_n.\end{aligned}\tag{2.17}$$

The quotient algebra $\mathbb{C}W^{e-} := \mathbb{C}\widetilde{W}^e / \langle z+1 \rangle$ of $\mathbb{C}\widetilde{W}^e$ by the ideal generated by $z+1$ will be called the *spin extended affine Weyl group algebra* associated to W . Denote by t_i (respectively, by $t_{\pi_r}^{\pm 1}$) element in $\mathbb{C}W^{e-}$ the image of \tilde{t}_i (respectively, $\tilde{\pi}_r^{\pm 1}$). The *spin extended affine Weyl group algebra* $\mathbb{C}W^{e-}$ has the following uniform presentation: $\mathbb{C}W^{e-}$ is the algebra generated by t_i 's, and $t_{\pi_r}^{\pm 1}$ for $r \in O_W^*$ subject to the relations

$$(t_i t_j)^{m_{ij}} = (-1)^{m_{ij}+1} \equiv \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ -1, & \text{if } m_{ij} = 2, 4 \end{cases} \tag{2.18}$$

with the following additional relations depending on W , i.e, if $W = W_{A_{n-1}}$,

$$\begin{aligned}t_{\pi_1}^2 t_{n-1} &= t_1 t_{\pi_1}^2, \\ t_{\pi_1}^n t_i &= t_i t_{\pi_1}^n \quad (1 \leq i \leq n-1), \\ t_{\pi_1} t_i &= (-1)^{n-1} t_{i+1} t_{\pi_1} \quad (1 \leq i \leq n-2);\end{aligned}\tag{2.19}$$

if $W = W_{B_n}$,

$$\begin{aligned}t_{\pi_1}^2 &= 1, \\ t_{\pi_1} t_i &= -t_i t_{\pi_1} \quad (2 \leq i \leq n), \\ t_{\pi_1} t_1 t_{\pi_1} t_1 &= -t_1 t_{\pi_1} t_1 t_{\pi_1};\end{aligned}\tag{2.20}$$

if $W = W_{D_n}$ and n is odd,

$$\begin{aligned}t_{\pi_n}^4 &= -1, \\ t_{\pi_n}^2 t_{n-1} &= t_n t_{\pi_n}^2, \\ t_{\pi_n} t_i &= (-1)^{\frac{n-1}{2}} t_i t_{\pi_n} \quad (1 \leq i \leq n-2), \\ t_{\pi_n} t_n &= (-1)^{\frac{n-1}{2}} t_1 t_{\pi_n};\end{aligned}\tag{2.21}$$

If $W = W_{D_n}$ and n is even,

$$\begin{aligned}t_{\pi_1}^2 &= 1, \quad t_{\pi_n}^2 = (-1)^{\frac{n}{2}+1}, \\ t_{\pi_1} t_{\pi_n} &= -t_{\pi_n} t_{\pi_1} \quad t_{\pi_1} t_1 t_{\pi_1} = -t_{\pi_n} t_n t_{\pi_n}, \\ t_{\pi_1} t_i &= t_i t_{\pi_1} \quad (2 \leq i \leq n-2), \\ t_{\pi_n} t_i &= (-1)^{\frac{n}{2}} t_{n-i} t_{\pi_n} \quad (2 \leq i \leq n-2), \\ t_{\pi_1} t_{n-1} &= t_n t_{\pi_1}, \quad t_{\pi_n} t_1 = (-1)^{\frac{n}{2}} t_{n-1} t_{\pi_n}.\end{aligned}\tag{2.22}$$

The algebra $\mathbb{C}W^{e-}$ has a superalgebra (i.e. \mathbb{Z}_2 -graded) structure by letting each t_i be odd and t_{π_r} be either even or odd depending on W (motivated

by Theorem 3.3 below) as follows: $|t_{\pi_r}| \in \mathbb{Z}_2$, the homogenous degree of t_{π_r} , is set to be

$$|t_{\pi_1}| = \begin{cases} 0, & \text{if } W = W_{A_{2k}} \text{ or } W_{D_{2k}} \\ 1, & \text{if } W = W_{A_{2k+1}} \text{ or } W_{B_n}. \end{cases} \quad (2.23)$$

$$|t_{\pi_n}| = \begin{cases} k \pmod{2}, & \text{if } W_{D_n} = W_{D_{2k}} \\ k \pmod{2}, & \text{if } W_{D_n} = W_{D_{2k+1}}. \end{cases} \quad (2.24)$$

3. THE CLIFFORD ALGEBRA

In this section, we recall the definition of the Clifford algebra \mathcal{C}_n . We show that the group W^e acts as automorphisms on \mathcal{C}_n which leads to Theorem 3.3 below.

3.1. The Clifford algebra \mathcal{C}_n . Denote by $\mathfrak{h}^* = \mathbb{C}^n$ the standard (respectively, reflection) representation of the Weyl group W of type A_{n-1} (respectively, of type B_n or D_n). Let $\{\alpha_i\}$ be the set of simple roots for W . Note that \mathfrak{h}^* carries a W -invariant nondegenerate bilinear form $(-, -)$ such that $(\alpha_i, \alpha_j) = -2 \cos(\pi/m_{ij})$. It gives rise to an identification $\mathfrak{h}^* \cong \mathfrak{h}$ and also a bilinear form on \mathfrak{h} which will be again denoted by $(-, -)$.

Denote by \mathcal{C}_n the Clifford algebra associated to $(\mathfrak{h}^*, (-, -))$. We shall denote by $\{c_i\}$ the generators in \mathcal{C}_n corresponding to a standard orthonormal basis $\{e_i\}$ of \mathbb{C}^n and denote by $\{\beta_i\}$ the elements of \mathcal{C}_n corresponding to the simple roots $\{\alpha_i\}$ normalized with

$$\beta_i^2 = 1.$$

More explicitly, \mathcal{C}_n is generated by c_1, \dots, c_n subject to the relations

$$c_i^2 = 1, \quad c_i c_j = -c_j c_i \quad (i \neq j). \quad (3.1)$$

For type A_{n-1} , we have $\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1})$, $1 \leq i \leq n-1$. For type B_n , we have an additional $\beta_n = c_n$, and for type D_n , $\beta_n = \frac{1}{\sqrt{2}}(c_{n-1} + c_n)$.

The action of W on \mathfrak{h} and \mathfrak{h}^* preserves the bilinear form $(-, -)$ and thus it acts as automorphisms of the algebra \mathcal{C}_n . This gives rise to a semi-direct product $\mathcal{C}_n \rtimes \mathbb{C}W$. Moreover, the algebra $\mathcal{C}_n \rtimes \mathbb{C}W$ naturally inherits the superalgebra structure by letting elements in W be even and each c_i be odd.

3.2. An action on \mathcal{C}_n . We introduce the following elements in W . They will be used throughout the paper. Let $\sigma_1^A \in W_{A_{n-1}}$ be the cyclic permutation $(12 \dots n)$, $\sigma_1^D, \sigma_n^D \in W_{D_n}$, and $\sigma_1^B \in W_{B_n}$ be such that

$$\begin{aligned} \sigma_1^D \cdot e_i &= \begin{cases} -e_i, & \text{if } i = 1, n \\ e_i, & \text{if } i \neq 1, n. \end{cases} & \sigma_n^D \cdot e_i &= \begin{cases} -e_{n+1-i}, & \text{if } i \neq n \\ (-1)^{n-1} e_1, & \text{if } i = n. \end{cases} \\ \sigma_1^B \cdot e_i &= \begin{cases} -e_1, & \text{if } i = 1 \\ e_i, & \text{if } i \neq 1. \end{cases} \end{aligned}$$

Remark 3.1. We write σ_r , instead of σ_r^X where $X \in \{A, B, D\}$ when the context is clear.

Proposition 3.2. *The extended affine Weyl group W^e acts on \mathcal{C}_n by the following formulas: $w \cdot c = c^w$ and $\pi_r \cdot c = c^{\sigma_r}$ where $c \in \mathcal{C}_n, w \in W$ and $r \in O_W^*$.*

Proof. We know that W naturally acts on \mathcal{C}_n . Suppose $W = W_{A_{n-1}}$, then we have the followings: for $c \in \mathcal{C}_n$,

$$\pi_1^2 s_{n-1} \cdot c = \pi_1^2 \cdot c^{s_{n-1}} = c^{\sigma_1^2 s_{n-1}} = c^{s_1 \sigma_1^2} = s_1 \pi_1^2 \cdot c.$$

For $1 \leq i \leq n-1$,

$$\pi_1^n s_i \cdot c = \pi_1^n \cdot c^{s_i} = c^{\sigma_1^n s_i} = c^{s_1 \sigma_1^n} = s_i \pi_1^n \cdot c.$$

For $1 \leq i \leq n-2$,

$$\pi_1 s_i \cdot c = \pi_1 \cdot c^{s_i} = c^{\sigma_1 s_i} = c^{s_{i+1} \sigma_1} = s_{i+1} \pi_1 \cdot c.$$

Therefore, W^e acts on \mathcal{C}_n . For $W = W_{B_n}$ or $W = W_{D_n}$, the computation is similar to type A_{n-1} case, and can be verified easily. \square

Proposition 3.2 gives rise to a semi-direct product $\mathcal{C}_n \rtimes \mathbb{C}W^e$. Moreover, the algebra $\mathcal{C}_n \rtimes \mathbb{C}W^e$ naturally inherits the superalgebra structure by letting elements in W^e be even and each c_i be odd.

3.3. A superalgebra isomorphism. Given two superalgebras \mathcal{A} and \mathcal{B} , we view the tensor product of superalgebras $\mathcal{A} \otimes \mathcal{B}$ as a superalgebra with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} (aa' \otimes bb') \quad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}) \quad (3.2)$$

where $|b|$ denotes the \mathbb{Z}_2 -degree of b , etc. Also, we shall use short-hand notation ab for $(a \otimes b) \in \mathcal{A} \otimes \mathcal{B}$, $a = a \otimes 1$, and $b = 1 \otimes b$.

Theorem 3.3. *Let $W = W_{A_{n-1}}, W_{B_n}$, or W_{D_n} and $v_i = \frac{1}{\sqrt{2}}(c_i + c_{n+1-i})$ for $1 \leq i \leq n$. We have an isomorphism of superalgebras:*

$$\Phi : \mathcal{C}_n \rtimes \mathbb{C}W^e \xrightarrow{\simeq} \mathcal{C}_n \otimes \mathbb{C}W^e$$

which extends the identity map on \mathcal{C}_n , sends $s_i \mapsto -\sqrt{-1}\beta_i t_i$, and

$$\pi_1 \mapsto \begin{cases} \beta_1 \cdots \beta_{n-1} t_{\pi_1}, & \text{for type } A_{n-1} \\ -\sqrt{-1} c_1 t_{\pi_1}, & \text{for type } B_n \\ \sqrt{-1} c_1 c_n t_{\pi_1}, & \text{for type } D_{2k}, \end{cases}$$

$$\pi_n \mapsto \begin{cases} v_1 \cdots v_{\frac{n}{2}} t_{\pi_n}, & \text{for type } D_n = D_{2k} \\ c_1 v_1 \cdots v_{\frac{n-1}{2}} c_{\frac{n+1}{2}} t_{\pi_n}, & \text{for type } D_n = D_{2k+1}. \end{cases}$$

The inverse map Ψ is the extension of the identity map on $\mathbb{C}W$ which sends $t_i \mapsto \sqrt{-1}\beta_i s_i$ and

$$\begin{aligned}
t_{\pi_1} &\mapsto \begin{cases} \beta_{n-1} \cdots \beta_1 \pi_1, & \text{for type } A_{n-1} \\ \sqrt{-1} c_1 \pi_1, & \text{for type } B_n \\ -\sqrt{-1} c_n c_1 \pi_1, & \text{for type } D_n = D_{2k}, \end{cases} \\
t_{\pi_n} &\mapsto \begin{cases} v_{\frac{n}{2}} \cdots v_1 \pi_n, & \text{for type } D_n = D_{2k} \\ \frac{c_{n+1}}{2} v_{\frac{n-1}{2}} \cdots v_1 c_1 \pi_n, & \text{for type } D_n = D_{2k+1}. \end{cases}
\end{aligned}$$

Proof. By [KW1, Theorem 2.4], the map Φ (respectively, Ψ) preserves the relations not involving π_r (respectively, t_{π_r}) for $r \in O_W^*$. Below, we show that Φ (respectively, Ψ) preserves the relations involving π_r (respectively, t_{π_r}).

If $W = W_{B_n}$, we see that Φ preserves (2.10) as follows: for $2 \leq i \leq n$,

$$\begin{aligned}
\Phi(\pi_1^2) &= -c_1 t_{\pi_1} c_1 t_{\pi_1} = t_{\pi_1}^2 = \Phi(1). \\
\Phi(\pi_1 s_1 \pi_1 s_1) &= c_1 t_{\pi_1} \beta_1 t_1 c_1 t_{\pi_1} \beta_1 t_1 = c_1 \beta_1 c_1 \beta_1 t_{\pi_1} t_1 t_{\pi_1} t_1 \\
&= -\beta_1 c_1 \beta_1 c_1 t_1 t_{\pi_1} t_1 t_{\pi_1} = \Phi(s_1 \pi_1 s_1 \pi_1). \\
\Phi(\pi_1 s_i) &= -c_1 t_{\pi_1} \beta_i t_i = -\beta_i c_1 t_{\pi_1} t_i \\
&= -\beta_i t_i c_1 t_{\pi_1} = \Phi(s_i \pi_1).
\end{aligned}$$

The map Ψ preserves (2.20) as follows: for $2 \leq i \leq n$,

$$\begin{aligned}
\Psi(t_{\pi_1}^2) &= -c_1 \pi_1 c_1 \pi_1 = \pi_1^2 = \Psi(1). \\
\Psi(t_{\pi_1} t_1 t_{\pi_1} t_1) &= c_1 \pi_1 \beta_1 s_1 c_1 \pi_1 \beta_1 s_1 \\
&= c_1 (c_1 + c_2) \pi_1 s_1 c_1 (c_1 + c_2) \pi_1 s_1 = \Psi(-t_1 t_{\pi_1} t_1 t_{\pi_1}). \\
\Psi(t_{\pi_1} t_i) &= -c_1 \pi_1 \beta_i s_i = \beta_i c_1 \pi_1 s_i = \beta_i s_i c_1 \pi_1 \\
&= \Phi(-t_i t_{\pi_1}).
\end{aligned}$$

Therefore, Φ and Ψ are (super)algebra homomorphisms for $W = W_{B_n}$. We see that Ψ is an inverse map of Φ on the generators. Hence, Φ and Ψ are inverse algebra isomorphisms. For $W = W_{A_{n-1}}$ and $W = W_{D_n}$, the computation is similar, but lengthy, and hence will be skipped. \square

4. THE TRIGONOMETRIC DOUBLE AFFINE HECKE-CLIFFORD ALGEBRA

In this section, we introduce the trigonometric double affine Hecke-Clifford algebras, \ddot{H}_W^c , and then establish their PBW properties. For $W = W_{A_{n-1}}$, we show that \ddot{H}_W^c is isomorphic to $\ddot{\mathcal{H}}_{tr}^c$ defined in [W].

4.1. The algebras \ddot{H}_W^c of type A_{n-1} . Recall $\sigma_1^A \in W_{A_{n-1}}$ the cyclic permutation $(12 \dots n)$.

Definition 4.1. Let $u \in \mathbb{C}$, and $W = W_{A_{n-1}}$. The *trigonometric double affine Hecke-Clifford algebra*, denoted by \ddot{H}_W^c or $\ddot{H}_{A_{n-1}}^c$, is the algebra

generated by $\pi_1^{\pm 1}, s_1, \dots, s_{n-1}, x_i, c_i, (1 \leq i \leq n)$, subject to the relations (2.2–2.4), (3.1), and the following additional relations:

$$x_i x_j = x_j x_i \quad (\forall i, j) \quad (4.1)$$

$$x_i c_i = -c_i x_i, \quad x_i c_j = c_j x_i \quad (i \neq j) \quad (4.2)$$

$$\sigma c_i = c_{\sigma i} \sigma \quad (1 \leq i \leq n, \sigma \in S_n) \quad (4.3)$$

$$x_{i+1} s_i - s_i x_i = u(1 - c_{i+1} c_i) \quad (4.4)$$

$$x_j s_i = s_i x_j \quad (j \neq i, i+1) \quad (4.5)$$

$$\pi_1^2 s_{n-1} = s_1 \pi_1^2 \quad (4.6)$$

$$\pi_1 s_i = s_{i+1} \pi_1 \quad (1 \leq i \leq n-2) \quad (4.7)$$

$$\pi_1^n s_i = s_i \pi_1^n \quad (1 \leq i \leq n-1) \quad (4.8)$$

$$\pi_1 x_i = x_i^{\sigma_1^A} \pi_1, \quad \pi_1 c_i = c_i^{\sigma_1^A} \pi_1. \quad (4.9)$$

Remark 4.2. Without $\pi_1^{\pm 1}$, we arrive at the defining relations of the *degenerate affine Hecke-Clifford algebra* $\mathfrak{H}_{A_{n-1}}^c$ (see [Naz]).

4.2. PBW basis for $\ddot{H}_{A_{n-1}}^c$. Let us introduce the ring $\mathbb{C}[P] = \mathbb{C}[P_1^{\pm 1}, \dots, P_n^{\pm 1}]$. The group $W = S_n$ acts on the ring by the formula

$$P_i^w = P_{w(i)} \quad \forall w \in S_n.$$

Also recall the triangular decomposition of \mathfrak{H}_W^c in [KW1, Th. 3.4], namely $\mathfrak{H}_W^c \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}W$. We set

$$E := \text{Ind}_W^{\mathfrak{H}_W^c} \mathbb{C}[P] \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}[P].$$

Hence we obtain the representation $\Phi^{\text{aff}} : \mathfrak{H}_W^c \rightarrow \text{End}(E)$ where x_i and c_i act by left multiplication. Together with [KW1, Prop. 3.3], we have the followings: for $f \in \mathbb{C}[\mathfrak{h}^*]$, $c \in \mathbb{C}_n$, and $g \in \mathbb{C}[P]$, the generator $s_i \in \mathfrak{H}_W^c$ for $1 \leq i \leq n-1$ acts by

$$s_i \cdot (fcg) = f^{s_i} c^{s_i} g^{s_i} + \left(u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) cg. \quad (4.10)$$

Lemma 4.3. *Let $\sigma_1 := \sigma_1^A$. The representation $\Phi^{\text{aff}} : \mathfrak{H}_{A_{n-1}}^c \rightarrow \text{End}(E)$ extends to the representation $\Phi : \ddot{H}_{A_{n-1}}^c \rightarrow \text{End}(E)$ by letting $\pi_1 \cdot (fcg) = P_1 f^{\sigma_1} c^{\sigma_1} g^{\sigma_1}$ where $f \in \mathbb{C}[\mathfrak{h}^*]$, $c \in \mathbb{C}_n$, $g \in \mathbb{C}[P]$.*

Proof. We need to check that the relations in \ddot{H}_W^c are preserved under the map Φ . It is enough to check the relations involving only π_1 . Suppose $f \in \mathbb{C}[\mathfrak{h}^*]$, $c \in \mathbb{C}_n$, $g \in \mathbb{C}[P]$, then the map preserves the relation (4.6) as

follows:

$$\begin{aligned}
\pi_1^2 s_{n-1} \cdot (fcg) &= \pi_1^2 \cdot (f^{s_{n-1}} c^{s_{n-1}} g^{s_{n-1}}) \\
&\quad + \pi_1^2 \left(u \frac{f - f^{s_{n-1}}}{x_n - x_{n-1}} + u \frac{c_{n-1} c_n f - f^{s_{n-1}} c_{n-1} c_n}{x_n + x_{n-1}} \right) cg \\
&= P_1 P_2 \left(f^{s_1 \sigma_1^2} c^{s_1 \sigma_1^2} g^{s_1 \sigma_1^2} \right) \\
&\quad + P_1 P_2 \left(u \frac{f^{\sigma_1^2} - f^{s_1 \sigma_1^2}}{x_2 - x_1} + u \frac{c_1 c_2 f^{\sigma_1^2} - f^{s_1 \sigma_1^2} c_1 c_2}{x_2 + x_1} \right) c^{\sigma_1^2} g^{\sigma_1^2} \\
&= s_1 \pi_1^2 \cdot (fcg).
\end{aligned}$$

The map preserves the relation (4.7) as follows: for $1 \leq i \leq n-2$,

$$\begin{aligned}
\pi_1 s_i \cdot (fcg) &= \pi_1 \cdot \left(f^{s_i} c^{s_i} g^{s_i} + \left(u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) cg \right) \\
&= P_1 (f^{\sigma_1 s_i} c^{\sigma_1 s_i} g^{\sigma_1 s_i}) \\
&\quad + P_1 \left(u \frac{f^{\sigma_1} - f^{\sigma_1 s_i}}{x_{i+2} - x_{i+1}} + u \frac{c_{i+1} c_{i+2} f^{\sigma_1} - f^{\sigma_1 s_i} c_{i+1} c_{i+2}}{x_{i+2} + x_{i+1}} \right) c^{\sigma_1} g^{\sigma_1} \\
&= s_{i+1} \pi_1 \cdot (fcg).
\end{aligned}$$

The map preserves the relation (4.8) as follows: for $1 \leq i \leq n-1$,

$$\begin{aligned}
\pi_1^n s_i \cdot (fcg) &= \pi_1^n \cdot \left(f^{s_i} c^{s_i} g^{s_i} + \left(u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) cg \right) \\
&= P_1 P_2 \dots P_n (f^{s_i} c^{s_i} g^{s_i}) \\
&\quad + P_1 P_2 \dots P_n \left(u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) cg \\
&= s_i \pi_1^n \cdot (fcg).
\end{aligned}$$

The relation (4.9) can be easily checked. \square

Lemma 4.4. *Let $W = W_{A_{n-1}}$ and $\sigma_1 := \sigma_1^A$. The extended affine Weyl group W^e acts faithfully on the algebra $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$ where s_i for $1 \leq i \leq n-1$ acts naturally and diagonally, while π_1 acts as $\sigma_1 \otimes P_1 \sigma_1$.*

Proof. The group W acts diagonally on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$. As operators on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$, we have $\pi_1 s_i \equiv s_{i+1} \pi_1$ for $1 \leq i \leq n-2$, $\pi_1^n s_i \equiv s_i \pi_1^n$ for $1 \leq i \leq n-1$ and $\pi_1^2 s_{n-1} \equiv s_1 \pi_1^2$. Then W^e acts on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$.

Suppose $\tilde{w} = \pi_1^k s_{i_l} \dots s_{i_1}$ is a reduced expression with $0 \leq i_j \leq n-1$ where $s_0 = \pi_1 s_{n-1} \pi_1^{-1}$, and \tilde{w} acts trivially on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$. Then $\tilde{w} \cdot 1 = 1$. This implies that $\tilde{w} \in W$. Since W acts faithfully on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[P]$ this implies that $\tilde{w} = 1$. \square

Below we give a proof of the PBW basis theorem for \check{H}_W^ϵ using in effect the extension of the representation $\mathfrak{H}_W^\epsilon \rightarrow \text{End}(E)$ to the representation $\check{H}_W^\epsilon \rightarrow \text{End}(E)$. The argument here will be adapted in the proof of Theorem 4.13.

Theorem 4.5. *Let $W = W_{A_{n-1}}$. The set $\{x^\alpha c^\beta w \mid \alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_2^n, w \in W^e\}$ forms a \mathbb{C} -linear basis for the \ddot{H}_W^ϵ . Equivalently, the multiplication of subalgebras $\mathbb{C}[\mathfrak{h}^*]$, \mathbb{C}_n , and $\mathbb{C}W^e$ induces a vector space isomorphism*

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}W^e \xrightarrow{\sim} \ddot{H}_W^\epsilon.$$

Proof. We show that the elements $x^\alpha c^\beta w$ viewed as operators on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}[P]$ are linearly independent.

For $\alpha = (a_1, \dots, a_n)$, $\nu = (b_1, \dots, b_n)$, we denote $|\alpha| = a_1 + \dots + a_n$ and $|\nu| = b_1 + \dots + b_n$. Define a Lexicographic ordering $<$ on the monomials $x^\alpha P^\nu$, $\alpha \in \mathbb{Z}_{\geq 0}^n$, $\nu \in \mathbb{Z}^n$, by declaring $x^\alpha P^\nu < x^{\alpha'} P^{\nu'}$, if $|\alpha| + |\nu| < |\alpha'| + |\nu'|$, or if $|\alpha| + |\nu| = |\alpha'| + |\nu'|$ then there exists an $1 \leq i \leq 2n$ such that $z_i < z'_i$ and $z_j = z'_j$ for each $j < i$, where $z_i = a_i$ if $1 \leq i \leq n$, and $z_i = b_{i-n}$ if $n+1 \leq i \leq 2n$.

Suppose that $S := \sum a_{\alpha\beta w} x^\alpha c^\beta w = 0$ for a finite sum over α, β, w and that some coefficient $a_{\alpha\beta w} \neq 0$; we fix one such β . Now consider the action S on an element of the form $x_1^{N_1} x_2^{N_2} \dots x_n^{N_n}$ for $N_n \gg \dots \gg N_1 \gg 0$. By Lemma 4.4, there exists a unique \tilde{w} such that the leading term of $\tilde{w} \cdot x_1^{N_1} x_2^{N_2} \dots x_n^{N_n}$, namely $(x_1^{N_1} x_2^{N_2} \dots x_n^{N_n})^{\tilde{w}}$, is maximal among all possible w with $a_{\alpha\beta w} \neq 0$ for some α . We note that $(x_1^{N_1} x_2^{N_2} \dots x_n^{N_n})^{\tilde{w}} = (x_1^{N_1} x_2^{N_2} \dots x_n^{N_n})^\sigma P^\lambda$ for some $\sigma \in W$ and $\lambda \in \mathbb{Z}^n$. Let $\tilde{\alpha}$ be chosen among all α with $a_{\alpha\beta\tilde{w}} \neq 0$ such that the monomial $x^{\tilde{\alpha}} (x_1^{N_1} x_2^{N_2} \dots x_n^{N_n})^{\tilde{w}} c^\beta$ appears as a maximal term with coefficient $\pm a_{\tilde{\alpha}\beta\tilde{w}}$. It follows from $S = 0$ that $a_{\tilde{\alpha}\beta\tilde{w}} = 0$. This is a contradiction, and hence the elements $x^\alpha c^\beta w$ are linearly independent. \square

In the classical theory of the trigonometric double affine Hecke algebra (tDaHa), the algebras are equipped with two presentations. Originally, one defines the tDaHa to be the algebra generated by \mathfrak{h}^* and W^e with certain relations. To obtain the second presentation, one simply rewrite the generators of tDaHa in terms of \mathfrak{h}^* , W , and the weight lattice Y corresponding to W . As an analogue to the classical theory, we show below that the algebra $\ddot{\mathfrak{H}}_{tr}^\epsilon$ introduced in [W] is isomorphic to $\ddot{H}_{A_{n-1}}^\epsilon$.

First, we recall the algebra $\ddot{\mathfrak{H}}_{tr}^\epsilon$ is generated by $\mathbb{C}[\mathfrak{h}^*]$, \mathbb{C}_n , S_n , and $e^{\pm \epsilon_i}$ ($1 \leq i \leq n$), subject to the relations (4.2–4.5) and the following additional relations:

$$\begin{aligned} e^{\epsilon_i} e^{\epsilon_j} &= e^{\epsilon_j} e^{\epsilon_i}, & e^{\epsilon_i} e^{-\epsilon_i} &= 1 \\ w e^{\epsilon_i} &= e^{w \cdot \epsilon_i} w, & c_j e^{\epsilon_i} &= e^{\epsilon_i} c_j \quad (\forall w \in S_n, \forall i, j) \\ [x_i, e^\eta] &= u \sum_{k \neq i} \text{sgn}(k-i) \frac{e^\eta - e^{s_{ki}(\eta)}}{1 - e^{\text{sgn}(k-i) \cdot (\epsilon_k - \epsilon_i)}} (1 - c_i c_k) s_{ki}. \end{aligned}$$

The algebra $\ddot{\mathfrak{H}}_{tr}^\epsilon$ admits a natural superalgebra structure with c_i being odd and all other generators being even. Note that the subalgebra generated by e^{ϵ_i} ($1 \leq i \leq n$), denoted by $\mathbb{C}[Y]$ is identified with the group algebra of the weight lattice of type GL_n ; the subalgebra generated by e^{ϵ_i} ($1 \leq i \leq n$) and

S_n is identified with the group algebra of the extended affine Weyl group W^e of type GL_n .

Theorem 4.6. *Let $W = W_{A_{n-1}}$. We have an isomorphism of superalgebras:*

$$F : \check{\check{\mathfrak{H}}}_{tr}^\epsilon \xrightarrow{\cong} \check{\check{H}}_W^\epsilon$$

which is the identity map on \mathfrak{h}^* , S_n , \mathcal{C}_n , and sends $e^{\epsilon_i} \mapsto s_{i-1} \dots s_1 \pi_1 s_{n-1} \dots s_i$ for $1 \leq i \leq n$.

Proof. By Theorem 4.5 and [W, Prop. 5.4], it follows that $\check{\check{\mathfrak{H}}}_{tr}^\epsilon \cong \check{\check{H}}_W^\epsilon$ as vector spaces. By a long and tedious calculation, we show that the defining relations in $\check{\check{\mathfrak{H}}}_{tr}^\epsilon$ are preserved under the map F . Hence F is an isomorphism of superalgebras. \square

Remark 4.7. As a consequence of Theorem 4.5 and Theorem 4.6, the even center $\mathcal{Z}(\check{\check{H}}_W^\epsilon)$ contains $\mathbb{C}[Y]^W$ and $\mathbb{C}[x_1^2, \dots, x_n^2]^W$ as subalgebras. In particular, $\check{\check{H}}_W^\epsilon$ is module-finite over its even center. Alternately, this can be seen for $\check{\check{\mathfrak{H}}}_{tr}^\epsilon$ using the locally isomorphism relating to its rational counterpart.

4.3. The algebras $\check{\check{H}}_W^\epsilon$ of type D_n . Let $W = W_{D_n}$. The extended affine Weyl group W^e is characterized by n even or odd. So the definition of the trigonometric double affine Hecke-Clifford algebra $\check{\check{H}}_W^\epsilon$ of type D_n depends on either $n \in \mathbb{N}$ is even or odd.

Recall that the elements $\sigma_1^D, \sigma_n^D \in W$ are defined in Subsection 3.2.

Definition 4.8. Let $n \in \mathbb{N}$ be even, $u \in \mathbb{C}$, and $W = W_{D_n}$. The *trigonometric double affine Hecke-Clifford algebra* of type D_n , denoted by $\check{\check{H}}_W^\epsilon$ or $\check{\check{H}}_{D_n}^\epsilon$, is the algebra generated by and $\pi_1^{\pm 1}, \pi_n^{\pm 1}, x_i, c_i, s_i$, ($1 \leq i \leq n$) subject to the relations (2.2–2.6), (4.1–4.5), and additional relations:

$$s_n c_n = -c_{n-1} s_n, \quad s_n c_i = c_i s_n \quad (i \neq n-1, n) \quad (4.11)$$

$$s_n x_n = -x_{n-1} s_n - u(1 + c_{n-1} c_n) \quad (4.12)$$

$$s_n x_i = x_i s_n \quad (i \neq n-1, n) \quad (4.13)$$

$$\pi_1^2 = 1, \quad \pi_n^2 = 1 \quad (4.14)$$

$$\pi_1 \pi_n = \pi_n \pi_1, \quad \pi_1 s_1 \pi_1 = \pi_n s_n \pi_n \quad (4.15)$$

$$\pi_1 s_i = s_i \pi_1, \quad \pi_n s_i = s_{n-i} \pi_n \quad (2 \leq i \leq n-2) \quad (4.16)$$

$$\pi_1 s_{n-1} = s_n \pi_1, \quad \pi_n s_1 = s_{n-1} \pi_n \quad (4.17)$$

$$\pi_r x_i = x_i^{\sigma_r^D} \pi_r, \quad \pi_r c_i = c_i^{\sigma_r^D} \pi_r \quad (r = 1, n; 1 \leq i \leq n). \quad (4.18)$$

Definition 4.9. Let $n \in \mathbb{N}$ be odd, $u \in \mathbb{C}$, and $W = W_{D_n}$. The *trigonometric double affine Hecke-Clifford algebra* of type D_n , denoted by $\check{\check{H}}_W^\epsilon$ or $\check{\check{H}}_{D_n}^\epsilon$, is the algebra generated by $\pi_n^{\pm 1}, x_i, c_i, s_i$, ($1 \leq i \leq n$) subject to the

relations (2.2–2.6), (4.1–4.5), (4.11–4.13) and additional relations:

$$\begin{aligned}\pi_n^4 &= 1 \\ \pi_n s_i &= s_{n-i} \pi_n \quad (2 \leq i \leq n-2) \\ \pi_n^2 s_{n-1} &= s_n \pi_n^2 \\ \pi_n s_1 &= s_{n-1} \pi_n, \quad \pi_n s_n = s_1 \pi_n \\ \pi_n x_i &= x_i^{\sigma_n^D} \pi_n, \quad \pi_n c_i = c_i^{\sigma_n^D} \pi_n \quad (1 \leq i \leq n).\end{aligned}$$

Recall that the element $\sigma_1^B \in W_{B_n}$ is defined in Subsection 3.2.

4.4. The algebras \ddot{H}_W^ϵ of type B_n .

Definition 4.10. Let $u, v \in \mathbb{C}$, and let $W = W_{B_n}$. The *trigonometric double affine Hecke-Clifford algebra* of type B_n , denoted by \ddot{H}_W^ϵ or $\ddot{H}_{B_n}^\epsilon$, is the algebra generated by and $\pi_1^{\pm 1}, x_i, c_i, s_i$, ($1 \leq i \leq n$) subject to the relations (2.2–2.4), (2.7–2.8), (4.1–4.5), and additional relations:

$$s_n c_n = -c_n s_n, \quad s_n c_i = c_i s_n \quad (i \neq n) \quad (4.19)$$

$$s_n x_n = -x_n s_n - \sqrt{2}v \quad (4.20)$$

$$s_n x_i = x_i s_n \quad (i \neq n) \quad (4.21)$$

$$\pi_1^2 = 1 \quad (4.22)$$

$$\pi_1 s_i = s_i \pi_1 \quad (2 \leq i \leq n) \quad (4.23)$$

$$\pi_1 s_1 \pi_1 s_1 = s_1 \pi_1 s_1 \pi_1 \quad (4.24)$$

$$\pi_1 x_i = x_i^{\sigma_1^B} \pi_1, \quad \pi_1 c_i = c_i^{\sigma_1^B} \pi_1 \quad (1 \leq i \leq n). \quad (4.25)$$

Remark 4.11. Without $\pi_1^{\pm 1}$, we arrive at the defining relations of the *degenerate affine Hecke-Clifford algebra* \mathfrak{H}_W^ϵ , for $W = W_{B_n}$ or W_{D_n} (see [KW1]).

4.5. PBW basis for \ddot{H}_W^ϵ . In this subsection, we prove a PBW type result for the algebra \ddot{H}_W^ϵ for $W = W_{B_n}$ or $W = W_{D_n}$. We first make a suitable modification from Subsection 4.2. We introduce the ring $\mathbb{C}[P] := \mathbb{C}[P_1^{\pm 1/2}, \dots, P_n^{\pm 1/2}]$ where P_i formally corresponds to e^{ϵ_i} . S_n naturally acts on $\mathbb{C}[P]$ by the formula $(P_i^{1/2})^w = P_{w(i)}^{1/2}$. The action can be extended to an action of $W = W_{D_n}$ by letting

$$\begin{aligned}(P_n^{1/2})^{s_n} &= P_{n-1}^{-1/2}, \quad (P_{n-1}^{1/2})^{s_n} = P_n^{-1/2} \\ (P_i^{1/2})^{s_n} &= P_i^{1/2}, \quad (i \neq n-1, n).\end{aligned}$$

Also, the action of S_n on $\mathbb{C}[P]$ can be extended to an action of $W = W_{B_n}$ by letting

$$(P_n^{1/2})^{s_n} = P_n^{-1/2}, \quad (P_i^{1/2})^{s_n} = P_i^{1/2}, \quad (i \neq n).$$

Consider the representation $\Phi^{aff} : \mathfrak{H}_W^\epsilon \rightarrow \text{End}(E)$, namely

$$E := \text{Ind}_W^{\mathfrak{H}_W^\epsilon} \mathbb{C}[P] \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}[P].$$

Note that x_i and c_i act by left multiplication. The generators $s_i \in \mathfrak{H}_W^\epsilon$ for $W = W_{D_n}$ ($1 \leq i \leq n$) act by the same formula (4.10) for $1 \leq i \leq n-1$ and in addition by

$$s_n \cdot (fcg) = f^{s_n} c^{s_n} g^{s_n} - \left(u \frac{f - f^{s_n}}{x_n + x_{n-1}} - u \frac{c_{n-1} c_n f - f^{s_n} c_{n-1} c_n}{x_n - x_{n-1}} \right) cg.$$

The generators $s_i \in \mathfrak{H}_W^\epsilon$ for $W = W_{B_n}$ ($1 \leq i \leq n$) act by the same formula (4.10) for $1 \leq i \leq n-1$ and in addition by

$$s_n \cdot (fcg) = f^{s_n} c^{s_n} g^{s_n} - \sqrt{2} \left(v \frac{f - f^{s_n}}{2x_n} \right) cg.$$

For more details treatment on \mathfrak{H}_W^ϵ , consult [Naz, KW1]. The following lemma is a counterpart to Lemma 4.3.

Lemma 4.12. *Let $W = W_{D_n}$ or W_{B_n} . The representation $\Phi^{aff} : \mathfrak{H}_W^\epsilon \rightarrow \text{End}(E)$ extends to the representation $\Phi : \ddot{H}_W^\epsilon \rightarrow \text{End}(E)$ by the following formulas:*

$$\pi_r \cdot (fcg) = \begin{cases} P_1 f^{\sigma_1} c^{\sigma_1} g^{\sigma_1}, & \text{if } r = 1, \\ (P_1 \dots P_n)^{1/2} f^{\sigma_n} c^{\sigma_n} g^{\sigma_n}, & \text{if } r = n, \end{cases}$$

where $f \in \mathbb{C}[\mathfrak{h}^*]$, $c \in \mathbb{C}_n$, $g \in \mathbb{C}[P]$.

Proof. By a direct and lengthy computation, the action \mathfrak{H}_W^ϵ on $E \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}[P]$ naturally extends to an action of \ddot{H}_W^ϵ . We will verify a few relations in $\ddot{H}_{D_n}^\epsilon$ for n is even, and leave the rest to the reader.

The map preserves the relation (4.14) as follows:

$$\begin{aligned} \pi_1^2 \cdot (fcg) &= \pi_1 \cdot P_1 (fcg)^{\sigma_1} = 1 \cdot (fcg). \\ \pi_n^2 \cdot (fcg) &= \pi_n \cdot (P_1 \dots P_n)^{1/2} (fcg)^{\sigma_n} \\ &= (P_1 \dots P_n)^{1/2} (P_n^{-1} \dots P_1^{-1})^{1/2} fcg = 1 \cdot (fcg). \end{aligned}$$

The map preserves the relation (4.15) as follows: since $\sigma_1 \sigma_n = \sigma_n \sigma_1$, then $\pi_1 \pi_n \cdot fcg = \pi_n \pi_1 \cdot fcg$. We also have

$$\begin{aligned} \pi_1 s_1 \pi_1 \cdot fcg &= \pi_1 s_1 \cdot P_1 (fcg)^{\sigma_1} \\ &= \pi_1 \cdot P_2 (fcg)^{s_1 \sigma_1} \\ &\quad + u \pi_1 \cdot P_2 \left(\frac{f^{\sigma_1} - f^{s_1 \sigma_1}}{x_2 - x_1} + \frac{c_1 c_2 f^{\sigma_1} - f^{s_1 \sigma_1} c_1 c_2}{x_2 + x_1} \right) (cg)^{\sigma_1} \\ &= P_1 P_2 (fcg)^{\sigma_1 s_1 \sigma_1} \\ &\quad + u P_1 P_2 \left(\frac{f - f^{\sigma_1 s_1 \sigma_1}}{x_2 + x_1} + \frac{c_2 c_1 f - f^{\sigma_1 s_1 \sigma_1} c_2 c_1}{x_2 - x_1} \right) cg. \end{aligned}$$

$$\begin{aligned}
& \pi_n s_n \pi_n \cdot f c g \\
&= \pi_n s_n \cdot (P_1 \dots P_n)^{1/2} (f c g)^{\sigma_n} \\
&= \pi_n \cdot (P_1 \dots P_{n-2} P_{n-1}^{-1} P_n^{-1})^{1/2} (f c g)^{s_n \sigma_n} \\
&\quad - u \pi_n \cdot (P_1 \dots P_{n-2} P_{n-1}^{-1} P_n^{-1})^{1/2} \left(\frac{f^{\sigma_n} - f^{s_n \sigma_n}}{x_n + x_{n-1}} \right) (c g)^{\sigma_n} \\
&\quad + u \pi_n \cdot (P_1 \dots P_{n-2} P_{n-1}^{-1} P_n^{-1})^{1/2} \left(\frac{c_{n-1} c_n f^{\sigma_n} - f^{s_n \sigma_n} c_{n-1} c_n}{x_n - x_{n-1}} \right) (c g)^{\sigma_n} \\
&= P_1 P_2 (f c g)^{\sigma_n s_n \sigma_n} + u P_1 P_2 \left(u \frac{f - f^{\sigma_n s_n \sigma_n}}{x_1 + x_2} + \frac{c_2 c_1 f - f^{\sigma_n s_n \sigma_n} c_2 c_1}{-x_1 + x_2} \right) c g.
\end{aligned}$$

Since $\sigma_1 s_1 \sigma_1 = \sigma_n s_n \sigma_n$, then it follows that $\pi_1 s_1 \pi_1 \cdot f c g = \pi_n s_n \pi_n \cdot f c g$.

The map preserves the relation (4.16) as follows: for $2 \leq i \leq n-2$, we have

$$\begin{aligned}
\pi_n s_i \cdot f c g &= \pi_n \cdot (f c g)^{s_i} + \pi_n \cdot \left(u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) c g \\
&= (P_1 \dots P_{n-1} P_n)^{1/2} (f c g)^{\sigma_n s_i} \\
&\quad + u (P_1 \dots P_{n-1} P_n)^{1/2} \left(\frac{f^{\sigma_n} - f^{\sigma_n s_i}}{-x_{n-i} + x_{n+1-i}} \right) (c g)^{\sigma_n} \\
&\quad + u (P_1 \dots P_{n-1} P_n)^{1/2} \left(\frac{c_{n+1-i} c_{n-i} f - f^{\sigma_n s_i} c_{n+1-i} c_{n-i}}{-x_{n-i} - x_{n+1-i}} \right) (c g)^{\sigma_n} \\
&= s_{n-i} \pi_n \cdot f c g.
\end{aligned}$$

It is easy to verify $\pi_1 s_i = s_i \pi_1$ for $2 \leq i \leq n-2$. For the rest of the relations in \ddot{H}_W^c , the calculation is similar. \square

We have the following PBW basis theorem for \ddot{H}_W^c .

Theorem 4.13. *Let $W = W_{D_n}$ or $W = W_{B_n}$, the elements $\{x^\alpha c^\beta w | \alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_2^n, w \in W^e\}$ form a \mathbb{C} -linear basis for \ddot{H}_W^c (called a PBW basis). Equivalently, the multiplication of subalgebras $\mathbb{C}[\mathfrak{h}^*], \mathbb{C}_n$, and $\mathbb{C}W^e$ induces a vector space isomorphism*

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}W^e \xrightarrow{\simeq} \ddot{H}_W^c.$$

Proof. We show that the elements $x^\alpha c^\beta w$ viewed as operators on $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}_n \otimes \mathbb{C}[P]$ are linearly independent. It is clear that, for either $W = W_{D_n}$ or $W = W_{B_n}$, the elements $x^\alpha c^\beta w$ span \ddot{H}_W^c . It remains to show that they are linearly independent. We shall treat the W_{B_n} case in detail and skip the analogous W_{D_n} case.

To that end, we shall refer to the argument in the proof of Theorem 4.5 with suitable modification. First, we observe that for each $w \in W_{B_n}^e$, the leading term of $w \cdot (x_1^{N_1} \dots x_n^{N_n})$ is $(x_1^{N_1} \dots x_n^{N_n})^\gamma P^\lambda$ for some $\gamma \in W_{B_n}$ and $\lambda \in (\frac{\mathbb{Z}}{2})^n$. Similar to type A case, we choose \tilde{w} such that the leading term $(x_1^{N_1} x_2^{N_2} \dots x_n^{N_n})^{\tilde{w}} = (x_1^{N_1} \dots x_n^{N_n})^\gamma P^\lambda$ is maximal for some γ, λ . Write

$\gamma = ((\eta_1, \dots, \eta_n), \sigma) \in W_{B_n} = \{\pm 1\}^n \rtimes S_n$. Let $\tilde{\alpha}$ be chosen among all α with $a_{\alpha\beta\tilde{w}} \neq 0$ such that the monomial $x^{\tilde{\alpha}}(x_1^{N_1}x_2^{N_2}\dots x_n^{N_n})^{\tilde{w}}c^\beta$ appears as a maximal term with coefficient $\pm a_{\tilde{\alpha}\beta\tilde{w}}$. Note that \tilde{w} may now not be unique, but the λ, σ , and $\tilde{\alpha}$ are uniquely determined. Then, by the same argument on the vanishing of a maximal term, we obtain that

$$\begin{aligned} 0 &= \sum_{\tilde{w}} a_{\tilde{\alpha}\beta\tilde{w}} x^{\tilde{\alpha}}(x_1^{N_1}x_2^{N_2}\dots x_n^{N_n})^{\tilde{w}} \\ &= \sum_{\gamma} a_{\tilde{\alpha}\beta\tilde{w}} x^{\tilde{\alpha}}(x_1^{N_1}x_2^{N_2}\dots x_n^{N_n})^\gamma P^\lambda, \end{aligned}$$

and hence

$$\sum_{(\eta_1, \dots, \eta_n)} a_{\tilde{\alpha}\beta\tilde{w}} (-1)^{\sum_{i=1}^n \eta_i N_i} = 0.$$

By choosing N_1, \dots, N_n with different parities (2^n choices) and solving the 2^n linear equations, we see that all $a_{\tilde{\alpha}\beta\tilde{w}} = 0$. This can also be seen more explicitly by induction on n . By choosing N_n to be even and odd, we deduce that for a fixed η_n , $\sum_{(\eta_1, \dots, \eta_{n-1}) \in \{\pm 1\}^{n-1}} a_{\tilde{\alpha}\epsilon\tilde{w}} (-1)^{\sum_{i=1}^{n-1} \eta_i N_i} = 0$, which is the equation for $(n-1)$ x_i 's and the induction applies. \square

Corollary 4.14. *Let $W = W_{D_n}$ or $W = W_{B_n}$. The even center of \ddot{H}_W^ϵ contains $\mathbb{C}[x_1^2, \dots, x_n^2]^W$.*

Proof. Suppose $f \in \mathbb{C}[x_1^2, \dots, x_n^2]^W$. Then $\pi_r f = f^{\sigma_r} \pi_1 = f \pi_1$. By [KW1, Prop. 4.6], f commutes with the generators s_1, \dots, s_n and c_1, \dots, c_n . So, f is in the even center of \ddot{H}_W^ϵ . \square

5. TRIGONOMETRIC SPIN DOUBLE AFFINE HECKE ALGEBRAS

In this section we introduce the trigonometric spin double affine Hecke algebra \ddot{H}_W^- , and then establish its connections to the corresponding trigonometric double affine Hecke-Clifford algebras \ddot{H}_W^ϵ .

5.1. The skew-polynomial algebra. We shall denote by $\mathbb{C}[\xi_1, \dots, \xi_n]$ the \mathbb{C} -algebra generated by ξ_1, \dots, ξ_n subject to the relations

$$\xi_i \xi_j + \xi_j \xi_i = 0 \quad (i \neq j). \quad (5.1)$$

This is naturally a superalgebra by letting each ξ_i be odd. We will refer to this as the *skew-polynomial algebra* in n variables. This algebra has a linear basis given by $\xi^\alpha := \xi_1^{k_1} \dots \xi_n^{k_n}$ for $\alpha = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, and it contains the polynomial subalgebra $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]$.

5.2. The algebra \ddot{H}_W^- of type A_{n-1} . Let $W = W_{A_{n-1}}$. Recall that the spin extended affine Weyl group $\mathbb{C}W^{e-}$ associated to a Weyl group W is generated by t_1, \dots, t_{n-1} , and t_{π_1} subject to the relations as specified in Section 2.3

Definition 5.1. Let $u \in \mathbb{C}$. The *trigonometric spin double affine Hecke algebra* of type A_{n-1} , denoted by \ddot{H}_W^- or $\ddot{H}_{A_{n-1}}^-$, is the algebra generated by $\mathcal{C}[\xi_1, \dots, \xi_n]$ and $\mathbb{C}W^{e-}$ subject to the relations:

$$t_i \xi_i = -\xi_{i+1} t_i + u \quad (1 \leq i \leq n-1) \quad (5.2)$$

$$t_i \xi_j = -\xi_j t_i \quad (j \neq i, i+1) \quad (5.3)$$

$$t_{\pi_1} \xi_i = (-1)^{n-1} \xi_{i+1} t_{\pi_1} \quad (1 \leq i \leq n-1) \quad (5.4)$$

$$t_{\pi_1} \xi_n = (-1)^{n-1} \xi_1 t_{\pi_1}. \quad (5.5)$$

5.3. The algebra \ddot{H}_W^- of type D_n . Let $u \in \mathbb{C}$ and $W = W_{D_n}$. In this subsection we define the *trigonometric spin double affine Hecke algebra* associated to W , denoted by \ddot{H}_W^- or $\ddot{H}_{D_n}^-$, for both n is even or odd.

Definition 5.2. Let n be odd. The algebra \ddot{H}_W^- is generated by $\mathcal{C}[\xi_1, \dots, \xi_n]$ and $\mathbb{C}W^{e-}$ subject to the relations (5.2-5.3) and the following additional relations:

$$t_n \xi_n = -\xi_{n-1} t_n + u \quad (5.6)$$

$$t_n \xi_i = -\xi_i t_n \quad (i \neq n-1, n) \quad (5.7)$$

$$t_{\pi_n} \xi_i = (-1)^{\frac{n-1}{2}} \xi_{n+1-i} t_{\pi_n} \quad (1 \leq i \leq n). \quad (5.8)$$

Definition 5.3. Let n even, the algebra \ddot{H}_W^- is generated by $\mathcal{C}[\xi_1, \dots, \xi_n]$ and $\mathbb{C}W^{e-}$ subject to the relations (5.2-5.7) and the following additional relations:

$$t_{\pi_1} \xi_i = \xi_i t_{\pi_1} \quad (1 \leq i \leq n)$$

$$t_{\pi_n} \xi_i = (-1)^{\frac{n}{2}+1} \xi_{n+1-i} t_{\pi_n}.$$

5.4. The algebra \ddot{H}_W^- of type B_n .

Definition 5.4. Let $u, v \in \mathbb{C}$, and $W = W_{B_n}$. The *trigonometric spin double affine Hecke algebra* of type B_n , denoted by \ddot{H}_W^- or $\ddot{H}_{B_n}^-$, is the algebra generated by $\mathcal{C}[\xi_1, \dots, \xi_n]$ and $\mathbb{C}W^{e-}$ subject to the relations (5.2-5.3) and the following additional relations:

$$t_n \xi_n = -\xi_n t_n + v$$

$$t_n \xi_i = -\xi_i t_n \quad (i \neq n)$$

$$t_{\pi_1} \xi_i = -\xi_i t_{\pi_1} \quad (1 \leq i \leq n).$$

Sometimes, we will write $\ddot{H}_W^-(u, v)$ or $\ddot{H}_{B_n}^-(u, v)$ for \ddot{H}_W^- or $\ddot{H}_{B_n}^-$ to indicate the dependence on the parameters u, v .

Remark 5.5. The algebra \ddot{H}_W^- is naturally a superalgebra with all t_i 's and ξ_i being odd generators. A generator t_{π_r} can be either even or odd depending on r and W , its degree is given by (2.23) and (2.24). Without t_{π_r} , we arrive at the defining relation of *degenerate spin affine Hecke algebra*, denoted by \mathfrak{H}_W^- , see [W, KW1].

5.5. A superalgebra isomorphism.

Theorem 5.6. *Let $W = W_{A_{n-1}}, W_{D_n}$ or W_{B_n} . Then,*

- (1) *there exists an isomorphism of superalgebras*

$$\Phi : \ddot{H}_W^{\mathfrak{c}} \longrightarrow \mathbb{C}_n \otimes \ddot{H}_W^-$$

which extends the isomorphism $\Phi : \mathbb{C}_n \rtimes \mathbb{C}W^e \longrightarrow \mathbb{C}_n \otimes \mathbb{C}W^{e-}$ (in Theorem 3.3) and sends $x_i \mapsto \sqrt{-2}c_i\xi_i$ for each i ;

- (2) *the inverse $\Psi : \mathbb{C}_n \otimes \ddot{H}_W^- \longrightarrow \ddot{H}_W^{\mathfrak{c}}$ extends $\Psi : \mathbb{C}_n \otimes \mathbb{C}W^{e-} \longrightarrow \mathbb{C}_n \rtimes \mathbb{C}W^e$ (in Theorem 3.3) and sends $\xi_i \mapsto \frac{1}{\sqrt{-2}}c_ix_i$ for each i .*

Proof. We need to show that Φ (respectively, Ψ) preserves the defining relations in $\ddot{H}_W^{\mathfrak{c}}$ (respectively, in \ddot{H}_W^-). By [KW1, Theorem 4.4], the map Φ (respectively, Ψ) preserves the relations not involving only π_r (respectively, t_{π_r}). So it is left to show that Φ (respectively, Ψ) preserves the defining relations which involve π_r and x_i 's (respectively, t_{π_r} or ξ_i 's) for $1 \leq i \leq n$. We verify relations for type B_n case below. For other relations and types, the computation is similar, and will be skipped.

For $W = W_{B_n}$ and $2 \leq i \leq n$, we have

$$\begin{aligned} \Phi(\pi_1 x_i) &= -\sqrt{2}c_1 t_{\pi_1} c_i \xi_i = -\sqrt{2}c_i c_1 t_{\pi_1} \xi_i = \Phi(x_i \pi_1). \\ \Phi(\pi_1 x_1) &= -\sqrt{2}c_1 t_{\pi_1} c_1 \xi_1 = \sqrt{2}t_{\pi_1} \xi_1 = -\sqrt{2}\xi_1 t_{\pi_1} = \sqrt{2}c_1 \xi_1 c_1 t_{\pi_1} \\ &= \Phi(-x_1 \pi_1). \end{aligned}$$

Thus Φ is a homomorphism of (super)algebras. Similarly, Ψ is a superalgebra homomorphism by the followings: for $1 \leq i \leq n$

$$\Psi(t_{\pi_1} \xi_i) = \frac{1}{\sqrt{2}}c_1 \pi_1 c_i x_i = -\frac{1}{\sqrt{2}}c_i x_i c_1 \pi_1 = \Psi(-\xi_i t_{\pi_1}).$$

Since Φ and Ψ are inverses on generators and hence they are indeed (inverse) isomorphisms. \square

5.6. PBW basis for \ddot{H}_W^- . We have the following PBW basis theorem for \mathfrak{H}_W^- .

Theorem 5.7. *Let $W = W_{A_{n-1}}, W_{D_n}$ or W_{B_n} . The multiplication of the subalgebras $\mathbb{C}W^-$ and $\mathbb{C}[\xi_1, \dots, \xi_n]$ induces a vector space isomorphism*

$$\mathbb{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}W^{e-} \xrightarrow{\cong} \ddot{H}_W^-.$$

Proof. It follows from the definition that \ddot{H}_W^- is spanned by the elements of the form $\xi^\alpha \sigma$ where σ runs over a basis for $\mathbb{C}W^{e-}$ and $\alpha \in \mathbb{Z}_+^n$. By Theorem 5.6, we have an isomorphism $\Psi : \mathbb{C}_n \otimes \ddot{H}_W^- \longrightarrow \ddot{H}_W^{\mathfrak{c}}$. Observe that the image $\Psi(\xi^\alpha \sigma)$ are linearly independent in $\ddot{H}_W^{\mathfrak{c}}$ by the PBW basis Theorems 4.5 and 4.13 for $\ddot{H}_W^{\mathfrak{c}}$. Hence the elements $\xi^\alpha \sigma$ are linearly independent in \ddot{H}_W^- . \square

Remark 5.8. \ddot{H}_W^- contains the skew-polynomial algebra $\mathcal{C}[\xi_1, \dots, \xi_n]$ and the spin extended affine Weyl group algebra $\mathbb{C}W^{e-}$ as subalgebras.

Remark 5.9. As a counterpart of $\ddot{\mathfrak{H}}_{tr}^c$, [W] also introduce the trigonometric spin double affine Hecke-Clifford algebra of type A , denoted by $\ddot{\mathfrak{H}}_{tr}^-$. As a consequence of Theorems 4.6 and 5.6, we have $\ddot{H}_{A_{n-1}}^- \cong \ddot{\mathfrak{H}}_{tr}^-$, and therefore $\ddot{H}_{A_{n-1}}^-$ is module-finite over its even center. We also have the following.

Corollary 5.10. *Let $W = W_{A_{n-1}}, W_{B_n}$ or W_{D_n} . The even center for \ddot{H}_W^- contains $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W$.*

Proof. By the isomorphism $\Phi : \ddot{H}_W^c \rightarrow \mathcal{C}_n \otimes \ddot{H}_W^-$, we have

$$Z(\mathcal{C}_n \otimes \ddot{H}_W^-) = \Phi(Z(\ddot{H}_W^c)) \supseteq \Phi(\mathbb{C}[x_1^2, \dots, x_n^2]^W) = \mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W.$$

Thus, $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W \subseteq Z(\ddot{H}_W^-)$. \square

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